

DEBT AND CAPITAL ACCUMULATION IN A TWO-SECTOR MODEL WITH NON-TRADED GOODS*

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The paper examines optimal debt and capital accumulation for an open economy which faces an imperfect international capital market. The major finding is that the optimal pattern of debt and capital accumulation is affected by relative factor intensities. Factor intensities determine whether substitution or complementary relationships exist between debt and capital. The relationships between the country's net wealth and its two components financial and productive are also determined by factor intensities.

1. Introduction

The problems of debt and capital accumulation for a country engaging in international trade have been investigated recently by several economists. Hamada (1969) analyzed the optimal capital accumulation policy of an open economy which faces an imperfect international capital market. Bardhan (1970) investigated the optimal policy of capital accumulation, focusing his entire analysis on the behavior over time of the country's trade deficit. His model describes an economy consisting of consumption and investment sectors; it faces a given world price of capital goods and a given rate of interest, whereas the world price of the consumption good can be affected by the optimal policy.

Fischer and Frenkel (1972) investigated a descriptive two-sector model with two traded goods and perfect international capital markets; the model incorporated the costs of adjustment associated with new investment. One of their major findings is that the time pattern of the major international accounts do not depend on factor intensities in production.

Bruno (1970) was the first to consider the optimal debt policy of a small-size growing economy with non-traded goods. The country faced imperfect capital markets internationally; Bruno investigated in detail the optimal pattern over time of the real exchange rate.

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In this paper we develop an international trade model with two sectors: the traded sector producing an investment good, and the non-traded sector producing a consumption good. The economy faces given international prices of the traded good, and imperfect international capital markets. We analyze the optimal debt and capital accumulation policy for such a country.

Our major finding, summarized in two propositions (section 4), is that the optimal pattern of debt and capital accumulation is affected by relative factor intensities. Factor intensities determine whether substitution or complementary relationships exist between debt and capital. The relationships between the country's net wealth and its two components – financial and productive – are also determined by factor intensities.

The plan of the paper is as follows: Section 2 presents the trade model. In section 3 we characterize the optimal policies for debt and capital accumulation. Section 4 presents and discusses the main results of the paper. The last section, section 5, which includes a summary is followed by an appendix.

2. The model

Assume an economy with two productive sectors: one producing for consumption purposes and the other producing for investment purposes. The output of the consumption good, denoted by H , is not traded internationally. The output of the investment good, denoted by M , is internationally tradable at a constant price, since the country is a relatively small buyer-supplier of this good. There are two primary factors of production: capital and labor, whose quantities are denoted by K and L , respectively.

The technology of production is specified by a transformation function which describes the production possibilities frontier

$$M = F(H, K, L), \quad (1)$$

In general, the transformation curve is derived from the separate production functions for each kind of output for given overall amounts of K and L . However, it also allows for the possibility of joint production and sectoral interdependencies in production.

Assume that production is subject to constant returns to scale and to positive and diminishing marginal productivities. We are thus permitted to rewrite (1) expressing all variables in per-capita units, denoted by lower case letter as follows:¹

$$m = f(h, k), \quad (2)$$

where

$$m = M/L, \quad h = H/L, \quad k = K/L.$$

¹The terms labor and population are used interchangeably throughout the analysis because the labor-leisure choice is assumed away in the model.

The transformation function must have the following properties (in the case of non-joint production):

$$-\infty < f_h < 0, \quad 0 < f_k < \infty; \quad -\infty < f_{hh}, \quad f_{kk} < 0; \\ f_{hh}f_{kk} - f_{hk}^2 > 0, \quad \text{for } 0 < h, \quad k < \infty; \quad (3a)$$

$$f_k(h, 0) = \infty, \quad f_k(h, \infty) = 0, \quad \text{for } h > 0; \quad (3b)$$

$$g(h, \infty) = -\infty, \quad \text{for all } h \geq 0; \quad (3c)$$

where

$$g(h, k) = f(h, k) - nk.$$

We assume that (3) also holds in cases of joint production. If technology is non-joint, the sign of the second-order partial derivative of the transformation function f_{hk} is determined by relative factor intensities according to (3').

(i) If good H is relatively labor-intensive, then

$$f_{hk} < 0.$$

(ii) If good H is relatively capital-intensive, then (3')

$$f_{hk} > 0.$$

The validity of (i) in (3') can be seen in the diagram in fig. 1. Two transformation curves are depicted, each corresponding to a different given level of k . A Rybczynski Line (R.L.) connects points A and B of equal slopes of the transformation curves. When good H is relatively more labor-intensive, B must lie to the left of A. By the concavity of the transformation function, the slope of the curve at point C (where the per-capita output of good H is the same as that at point A) must be higher in absolute values than the slope of the curve at point A.

The validity of (ii) in (3') can be similarly shown.

Separability of the transformation function (i.e., $f_{hk} = 0$) almost always means that outputs are produced jointly. The only case in which the underlying production structure can be portrayed by separate and independent production functions for each kind of output is the uninteresting case in which the two production functions are identical.²

Foreign capital can be borrowed or the country's capital can be lent abroad. A supply schedule linking the per-capita interest liabilities on foreign capital (denoted by R) to the per-capita amount of net borrowing (denoted by b) is given. We note that a positive value of b means a debt, whereas a negative value

²See Hail (1973).

of b means financial assets are held by the country abroad. Let the *marginal rate of interest* (given by the derivative of R) be a positive and increasing function of the level of debt,

$$R = R(b), \quad R_b > 0, \quad R_{bb} > 0, \quad \text{for } -\infty < b < +\infty, \quad (4)$$

$$R_b(-\infty) = 0, \quad R_b(+\infty) = \infty.$$

This reflects situations where the terms under which foreign loans are extended to the country get worse as debt increases with a given level of population, and get better as population increases (representing an increase in production capacity³) with a given overall level of debt.

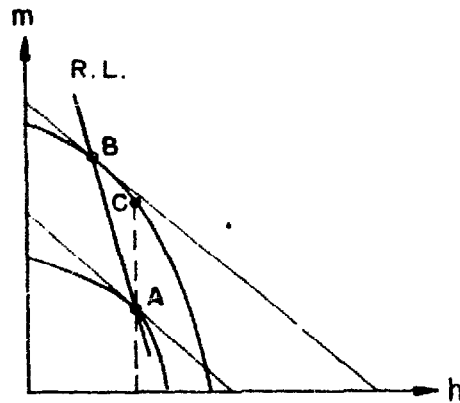


Fig. 1

Next, we assume a utility function, $U(\cdot)$, whose argument is the per-capita amount of the consumption good. The utility function is increasing and concave with respect to its argument,

$$U = U(h), \quad U_h > 0, \quad U_{hh} < 0. \quad (5)$$

Denote net exports (in per-capita units) by x so that when x is positive, exports exceed imports; conversely, a negative x means that the country's imports exceed its exports. Let the population be growing at a constant rate n . The change over time in the capital-labor ratio k is given by

$$\dot{k} = m - x - nk. \quad (6)$$

³Another possible formulation is to include in the interest liabilities' function productive capital as an additional argument. The present formulation was chosen for simplicity of exposition, however the same method of analysis applies in the alternative formulation.

Eq. (6) implies that the gross investment per capita ($k + nk$) increases with production of tradable goods (m) and decreases with net exports (x).

The change over time in per-capita debt is given by

$$\dot{b} = R - x - nb. \quad (7)$$

Eq. (7) implies that the gross increase over time in per-capita debt ($\dot{b} + nb$) increases with the amount of interest liabilities and decreases with net exports.

Assume that the length of the planning horizon is infinity. The intertemporal utility function is assumed to be time-additive without any time preference. Later, we discuss some of the implications of introducing a positive time preference into the model. The country evaluates different consumption programs by using the 'overtaking criterion'.⁴ According to this criterion, a feasible consumption program $h^a(t)$ is said to *overtake* another feasible consumption program $h^b(t)$ if there exists a T_0 such that for all $T > T_0$,

$$\int_{t=0}^T [U(h_s^a) - U(h_s^b)] ds > 0. \quad (8)$$

We are ready now to derive the optimum conditions. Similarly to solving a standard calculus of variations problem, we define λ_1 as an auxiliary variable associated with k and λ_2 as an auxiliary variable associated with b . We thus get necessary conditions (for an interior solution) as follows:

$$\lambda_1 = -\lambda_2 \equiv \lambda, \quad (9)$$

$$U_h + \lambda_1 f_h = 0, \quad (10)$$

$$\dot{\lambda}_1 = n\lambda_1 - \lambda_1 f_k, \quad (11)$$

$$\dot{\lambda}_2 = n\lambda_2 - R_b \lambda_2. \quad (12)$$

Using (9) to equate (11) and (12), we get

$$f_k = R_b. \quad (13)$$

Eqs. (10) and (13) are referred to as momentary optimal conditions, whereas (11) and (12) are referred to as dynamic conditions.

We first suggest an economic interpretation of (13). This describes an equality between the marginal productivity of capital in the tradable sector (f_k) and the marginal factor cost of debt (R_b). This equality is anticipated in our model, since

⁴See Koopmans (1967) for a discussion of this criterion. Note that it is a more discriminating criterion than maximizing the sum of utilities over the horizon.

at each moment of time the country can instantaneously and costlessly substitute productive capital for financial assets held abroad, or vice versa.

Eq. (10) can be interpreted as follows. An additional unit of h increases utility by U_h . This change requires reducing the level of production of the tradable good by f_h . This quantity is evaluated by $\lambda_1 f_h$, where λ_1 is the marginal utility of productive capital.

Dynamic equations such as (11)–(12) are well-known in economic growth. Eqs. (6)–(7) and (10)–(13), together with the initial conditions on b and k and the terminal conditions,

$$\lim_{t \rightarrow \infty} \lambda_1 k = \lim_{t \rightarrow \infty} \lambda_2 b < \infty,$$

constitute a set of necessary and sufficient conditions for optimum.

Using these relationships, in the next section we characterize the optimal dynamic path.

3. The optimum path

First, consider an optimal steady state where all variables, expressed in per-capita units, are constant over time (i.e., $\dot{k} = \dot{b} = \dot{\lambda} = 0$). Setting the left-hand sides of eqs. (6)–(7) and (11)–(12) equal to zero, we get

$$f(h^*, k^*) - x^* - nk^* = 0, \quad (14a)$$

$$R(b^*) - x^* - nb^* = 0, \quad (14b)$$

$$R_y(h^*) - n = 0, \quad (14c)$$

$$f_k(h^*, k^*) - n = 0. \quad (14d)$$

The set of four equations in (14), under familiar conditions,⁵ has a unique steady state solution for the four variables: h^* , x^* , k^* , b^* .⁶ Given those values and using (2), the steady state level of m^* is then uniquely determined.

We now attempt to characterize the optimal path, confining our analysis first to the case where the transformation function is not separable (i.e., $f_{hk} \neq 0$). A separate analysis of the separable case will be done later.

It is of interest to describe the optimum path in the (b, k) plane. Note that since k and b can be instantaneously substituted for each other, initial conditions are given by the difference $k_0 - b_0$. Putting this differently, all points on a 45° line

⁵See part I of the appendix.

⁶Note that b^* may be either positive or negative depending upon the function $R(b)$ and the value of n , x^* however is always negative.

passing through (b_0^*, k_0) describe states which can be reached instantaneously in the initial period. Let us express all the variables as functions of (b, k) . From eqs. (10) and (13) we get the functions $h(b, k)$ and $\lambda(b, k)$ with the following derivatives:

$$\begin{aligned} h_k &= -f_{kk}/f_{hk}, & h_b &= -R_{bb}/f_{hk}; \\ \lambda_k &= [f_{kk}U_{hh} + \lambda(f_{hh}f_{kk} - f_{hk}^2)]/f_h f_{hk}, \\ \lambda_b &= [-R_{bb}(U_{hh} + \lambda f_{hh})]/f_h f_{hk}. \end{aligned} \tag{15}$$

Differentiating $\lambda(b, k)$ with respect to time, substituting the right-hand sides of eqs. (6)–(7) into the result, and equating to (12), we get the function $x(b, k)$ in an implicit form as follows:

$$\lambda_k(m - x - nk) + \lambda_b(-x + R - nb) - \lambda(n - R_b) = 0. \tag{16}$$

We now turn to dynamic properties of the optimal path. First, consider the curve along which $k = 0$. From (6) and (16) we eliminate x by setting $k = 0$ to get

$$k = 0 \Rightarrow \lambda(R_b - n) + \lambda_b(R - m - n(b - k)) = 0. \tag{17}$$

Second, we consider the curve along which $b = 0$. From (7) and (16) we eliminate x by setting $b = 0$ to get

$$b = 0 \Rightarrow \lambda(R_b - n) - \lambda_k(R - m - n(b - k)) = 0. \tag{18}$$

Eqs. (17) and (18) describe two implicit functions of k and b . In general, the expressions for the total differentials of these two functions involve third-order partial derivatives of the functions of utility, transformation and interest liabilities. The economic theory, however, does not give us any guide as to their magnitudes. This problem is avoided when we restrict the dynamic analysis to the neighborhood of the steady state.

Totally differentiating (17) and (18) and evaluating the expressions at the steady state, we get

$$\left[\frac{dk}{db} \right]_{k=0} = - \frac{(\lambda R_{bb}/\lambda_b) + (f_h R_{bb}/f_{hk})}{f_h f_{kk}/f_{hk}}, \tag{19}$$

$$\left[\frac{dk}{db} \right]_{b=0} = - \frac{-(\lambda R_{bb}/\lambda_k) + (f_h R_{bb}/f_{hk})}{f_h f_{kk}/f_{hk}}. \tag{20}$$

We derive in three steps the partial derivatives of k and b with respect to k (holding b constant), and evaluate them at the steady state. First, the expression for x_k is obtained by differentiating (16) with respect to k . Second, we differentiate the right-hand sides of (6) and (7) with respect to k . Third, we substitute x_k and h_k [from (15)] into these results to get

$$\frac{\partial k}{\partial k} = -\frac{f_h f_{kk} \lambda_b}{f_{hk}(\lambda_b + \lambda_k)}, \quad \dagger$$

$$\frac{\partial b}{\partial k} = \frac{f_h f_{kk} \lambda_k}{f_{hk}(\lambda_b + \lambda_k)}. \quad (21)$$

Armed with (19)–(21), we are able to describe optimal paths in phase diagrams. We must distinguish between two cases: a case in which $f_{hk} > 0$ and a case in which $f_{hk} < 0$. An economic interpretation is given below.

Case I: $f_{hk} > 0$

In this case, we get [from (2), (4), (5) and (15)] $\lambda_b < 0$ and $\lambda_k < 0$. Using these results, we observe from (19) that $[dk/db]_{k=0} > 0$ while the sign of $[dk/db]_{b=0}$ is not a priori determined. However, from (19)–(20) we get⁷

$$\left[\frac{dk}{db} \right]_{k=0} - \left[\frac{dk}{db} \right]_{b=0} = \frac{-\lambda R_{bb}([f_{hk}/\lambda_b] + [f_{hk}/\lambda_k])}{(f_h f_{kk})} > 0. \quad (22)$$

The signs of the expressions in (21) are given in this case by $\partial k/\partial k < 0$, $\partial b/\partial k > 0$.

There are only two types of phase diagrams in this case. They are depicted in figs. 2(a) and 2(b). The heavy arrows denote optimal paths.

Case II: $f_{hk} < 0$

In this case we get [from (2), (4), (5) and (15)] $\lambda_b > 0$ and $\lambda_k > 0$. Using these results, we observe from (19) that $[dk/db]_{k=0} > 0$ while the sign of $[dk/db]_{b=0}$ is not a priori determined.

The signs of the expression in (21) are given in this case by: $\partial k/\partial k > 0$ and $\partial b/\partial k < 0$. Using (22), there are only two types of phase diagrams in this case, depicted in figs. 3(a) and 3(b). The heavy arrows denote optimal paths.

We now turn to the case where the transformation function is separable (i.e., $f_{hk} = 0$). In this case, eq. (10) is an implicit function in h and λ , while

⁷Note that from (15) the term in parentheses in the numerator of (22) is negative regardless of the sign of f_{hk} .

eq. (13) is an implicit function in b and k . The relationship between k and b along the optimal path is denoted by Q in fig. 4.⁸

Q is a downward-sloping curve, since upon differentiating (13) we get

$$dk/db = R_{bb}/f_{kk} < 0. \tag{23}$$

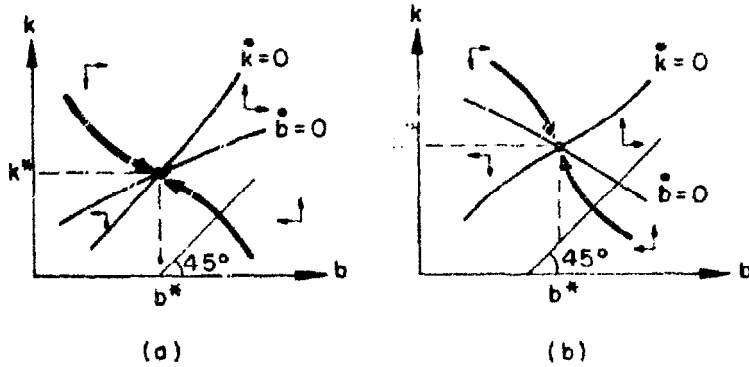


Fig. 2

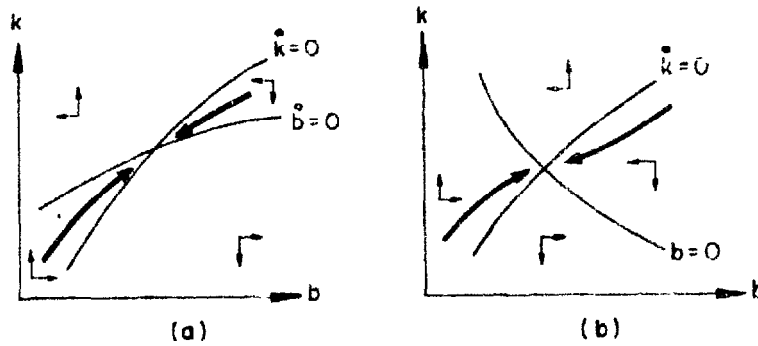


Fig. 3

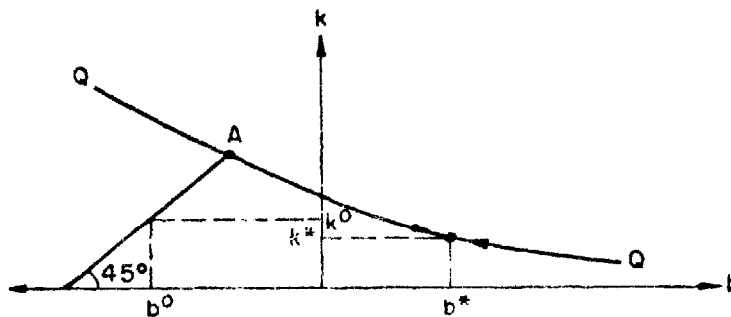


Fig. 4

Starting from the initial point (b_0, k_0) , the country (when behaving optimally) should move instantaneously along the 45°-line passing through this point until it reaches point A on the Q -curve. The dynamic movement of the country is then

⁸See Hochman et al. (1973) for an analysis of a similar model in a different economic context.

restricted to points on this curve. Note that the steady state (b^*, k^*) is a stable point on the Q -curve.⁹

4. Relationships between capital and debt

Let the difference between capital and debt be defined as *net wealth*, denoted by s (i.e., $s = k - b$). We say that debt and capital are complementary (substitutable) if they move in the same (opposite) directions along the optimum path. Let capital (k) and financial assets ($-b$) be defined as normal if they are positively related to net wealth. We say that these are *local* properties if confined to the neighborhood of the steady state, and *global* properties if valid everywhere.

We are now in a position to state the main results of the paper in the following two propositions.

Proposition 1. *If the non-traded consumption good is relatively capital-intensive (labor-intensive), then capital and debt are locally substitutable (locally complementary).*

Proof. If the non-traded consumption good is relatively capital-intensive (labor-intensive), then from (3') $f_{hk} > 0$ (< 0). From the discussion of case I (case II) in section 3 we get the result.

Proposition 2. *If the non-traded consumption good is relatively capital-intensive (labor-intensive), then capital and financial assets are globally normal (inferior).*

Proof. If the non-traded consumption good is relatively capital-intensive, then from (3') $f_{hk} > 0$. In section 3, case I, we dealt with local properties of the optimal path. In fig. 2 all points along a 45°-line represent the same net wealth, while points located below this line represent a lower level of net wealth. It is easily seen that the amounts of financial assets and net wealth move locally in the same directions. Furthermore, this is a global property. To show this, observe that the optimal path will never have a positive slope which is smaller than 45°. Otherwise, there exists a 45°-line intersecting the path at two points. Since the country can move instantaneously along 45°-lines in the (b, k) plane, between any two such points, it will choose the one which is closer to the steady state. This proves the proposition for the case where $f_{hk} > 0$. The other part is similarly proved.

Remarks

(a) Note that in all cases, net wealth changes monotonically with respect to time.

⁹For a proof see part II of the appendix.

(b) If the non-traded consumption good is relatively capital-intensive, then locally differentiating (16) and using (3a), we get

$$\partial x/\partial k < 0, \quad \partial x/\partial b > 0.$$

By Proposition 1, the level of net exports (x) changes monotonically with respect to time. Net exports will move locally in the same direction that net wealth does.

(c) If the non-traded consumption good is relatively labor-intensive, then upon differentiating (13) and using (3a), we get that

$$\partial h/\partial k < 0, \quad \partial h/\partial b < 0.$$

By Proposition 1 the level of consumption (h) changes monotonically with respect to time.

(d) If there exists a positive subjective rate of time preference, then the two propositions do not necessarily hold. If the rate of time preference is relatively high, then there is a possibility that no finite steady state exists – since debt may increase indefinitely.

Consider now an economic interpretation of Proposition 1.

If k is exogenously increased in the neighborhood of the steady state an immediate adjustment would take place along iso-wealth line. Capital goods would be sold and debt retired. There would be a blip in the trade account reflecting the large instantaneous sale of capital goods. In the case in which the non-traded good is capital-intensive, in subsequent periods $\dot{h} > 0$, $\dot{k} < 0$. This comes about through the following mechanism: with reduced b , R is lower, implying a smaller MP_k which implies a larger capital-labor ratio in each industry. This comes about by a contraction of capital-intensive industry, i.e., home goods. Even though the investment good industry is expanded, the capital stock cannot be maintained by the albeit increased rate of production. Thus $\dot{k} < 0$, R rises, pulling up MP_k . This shifts factors from investment goods production to home goods, and $\dot{k} < 0$ until we are back at the steady state.

5. Summary

The relationship between the major international trade accounts of a country and its technology is in general, complex and its dynamic analysis requires the use of mathematical techniques. In this paper the analysis of a model for a small-size growing economy with non-traded goods focuses on the relationships between the optimal debt and capital accumulation policy and the country's technology. It is shown that factor intensities determine the pattern of debt and capital accumulation.

A further analysis is needed, however, in order to establish whether or not the results of this paper carry over to different international trade models and thus have wider applicability.

Appendix

(1) Existence and uniqueness of the steady state

Beginning with some preliminaries, let x_0 be defined by (A.1) as follows:

$$x_0 = g(0, 0) = f(0, 0), \quad (\text{A.1})$$

where $g(h, k)$ is defined as in (3).

Let the family of functions $G(x)$ be defined by

$$G(x): g(h, k) = x. \quad (\text{A.2})$$

Note that $G(x_0)$ passes through the origin in the (h, k) plane. From (3) we know that in a neighbourhood of the origin h will increase with k along $G(x_0)$. The conditions in (3) also imply that $G(x_0)$ will cross the horizontal axis again for a sufficiently large value of k . The (non-empty) family of functions $G(x)$ is described in fig. 5.

Lemma 1. Let $G(x_1) \in G(x)$ be defined over a segment of the k -axis. Then, $G(x_1)$ has at most one extremum point in h . At this point, if it exists, h attains its highest level in $G(x_1)$.

Proof. Differentiate $G(x_1)$ to get

$$[dh/dk]_{G(x_1)} = -(f_k - n)/f_h. \quad (\text{A.3})$$

If (A.3) vanishes on the segment of the k -axis on which $G(x_1)$ is defined then $G(x_1)$ has an extremum, if not then no extremum exists. Assume that at least one extremum exists then by twice differentiating $G(x)$ we get

$$[d^2h/dk^2]_{G(x_1)} = \frac{f_{kk}}{f_h} + \frac{f_{hk} + f_{hh}(dh/dk)}{f_h} \left[\frac{dh}{dk} \right]_{G(x_1)}. \quad (\text{A.4})$$

Thus, at any point where $[dh/dk]$ vanishes we get

$$[d^2h/dk^2] < 0, \quad (\text{A.5})$$

which implies that at this point h attains a maximum. Furthermore, by (A.5) the maximum point is unique. This proves Lemma 1.

From (3b) $G(x_0)$ has a maximum. In fig. 5 $G(x_0)$ crosses the horizontal axis at $(0, \bar{k})$ and has a peak at $(h_{x_0}^*, k_{x_0}^*)$. The shape of the function $G(0)$, if $x_0 > 0$, is similar to $G(x_0)$. It passes through $(\frac{1}{2}, 0)$ and has a single peak at (h_0^*, k_0^*) . Due to (3b) and (3c) $h_0^* > \frac{1}{2}$ and $k_0^* > 0$ and the two curves can never intersect each other. Consider, now, the function $G(x_2)$ for $x_2 < 0$. It is defined only on the segment of the k -axis such that

$$-x_2/n \leq k \leq \bar{k}_{x_2}, \tag{A.6}$$

where \bar{k}_{x_2} solves $g(0, \bar{k}_{x_2}) = x_2$.

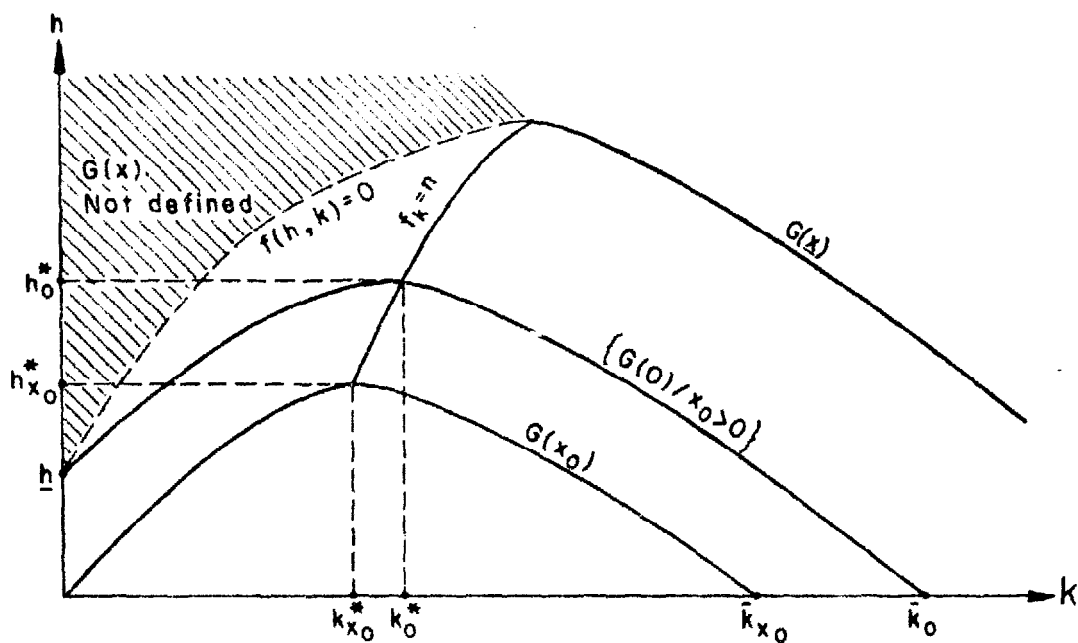


Fig. 5

Let x be the lower bound of all $x < 0$ for which there exist an $h > 0$ so that (h, x) fulfill the following equations:

$$\begin{aligned} f(h, -x/n) &= 0, \\ f_k(h, -x/n) &= n. \end{aligned} \tag{A.7}$$

Then

$$-\infty \leq x < 0. \tag{A.8}$$

Note that the locus of points in the (h, k) plane which fulfill $f(h, k) = n$ for $h_0^* \leq h \leq \bar{h}$ where \bar{h} fulfills $f(\bar{h}, -x/n) = 0$, is connected.

Lemma 2. Let $G(x): g(h, k) = x$, be defined for some $\underline{x} < x \leq 0$. Then the curve $f_k(h, k) - n = 0$ intersects $G(x)$ at its peak and only there.

Proof. At a peak of $G(x)$ the derivative of $G(x)$ vanishes. From (A.3), and since $f_h < 0$ for all $h > 0$, this implies that $f_k - n = 0$. Therefore the curve $f_k(h, k) - n = 0$ passes through the peak of $G(x)$. Since by Lemma 1 there is only one type of extremum for $G(x)$, the line $f_k - n = 0$ cannot pass through points other than the peak of $G(x)$. This proves Lemma 2.

For any given level of n consider, now, implicit functions of b and x described in the following equations:

$$R_b(b) = n, \quad (\text{A.9a})$$

$$R(b) - nb = x. \quad (\text{A.9b})$$

Let $x(n)$ be a solution of (A.9), then

$$\hat{x} \leq x \leq 0, \quad \text{for } -\infty \leq n \leq \infty,$$

where $-\infty \leq \hat{x} < 0$, is the lower bound of all x solving (A.9).

Note that $x = 0$ for $n = R_b(0)$ and that for every $n_r < R_b(0)$ there is an $n_1 > R_b(0)$ which fulfill $x(n_0) = x(n_1)$. Also note the $x(n)$ is monotonous in the segments $0 \leq n \leq R_b(0)$ and $R_b(0) \leq n < \infty$. If $\chi(\eta)$ fulfill $\max(\bar{x}, \hat{x}) < x(n) < 0$, then n is admissible. Let \underline{n} be the lower bound of all admissible n and \bar{n} the upper bound. Then

$$0 \leq \underline{n} < R_b(0) < \bar{n} \leq \infty. \quad (\text{A.10})$$

We can prove now the existence and uniqueness theorem.

Theorem 3. If the population rate of growth, n , lies between \underline{n} and \bar{n} then there exists a unique steady state for the dynamic model [eqs. (1)-(12)].

Proof. Let (A.11) be a steady state.

$$R_b(b^*) = n, \quad (\text{A.11a})$$

$$R(b^*) - nb^* = x^*, \quad (\text{A.11b})$$

$$f_k(h^*, k^*) = n, \quad (\text{A.11c})$$

$$f(h^*, k^*) - nk^* = x^*. \quad (\text{A.11d})$$

The values b^* and x^* can be solved uniquely from (A.11a) and (A.11b), respectively. From the construction of \underline{n} and \bar{n} we know that $\underline{x} < x^* < \bar{x}$. Thus, from Lemma 1 and the discussion following it, eq. (A.11d) has a single peak in the (h, k) plane. According to Lemma 2, (A.11c) passes through the peak of the $G(x^*)$ curve and only through it. This point of intersection yields (h^*, k^*) .

Q.E.D.

An interesting point is what happens if $n \ll \underline{n}$ or if $n \gg \bar{n}$. Without attempting to be rigorous, it seems that in the first case the situation is that the economy increases total investment per capita to infinity. When $n \gg \bar{n}$ the economy is trapped and is headed towards bankruptcy.

To illustrate Theorem 3 consider a simple case $f(h, k) = (a + bk^\alpha - ch^\beta)^\delta$, where $0 < \delta, \alpha < 1, \beta > 1, a, b, c > 0$ are constants. It is readily verified that this function satisfies (3) and yields a unique solution for (A.11c) and (A.11d) for every x^* .

(II)

In order to show that the steady state in the case $f_{hk} = 0$ is stable it is convenient to analyze the (k, λ) -plane in fig. 6.

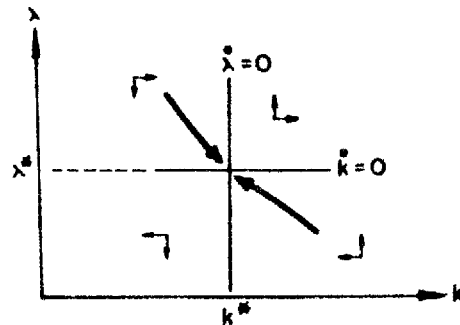


Fig. 6

The behavior of economy along the optimal path designated by heavy arrows in the (λ, k) -plane implies that the steady state is stable.

The construction of the phase diagram in fig. 6 is as follows: From (11) we get

$$\dot{\lambda} = 0 \Rightarrow f_k(k^*) = n,$$

determining uniquely k^* .

Since there is a one-to-one functional relationship between b and k we must have $b = 0 \Leftrightarrow k = 0$. Using (6) and (7) this implies

$$f(h, k) - nk = R - nb. \tag{A.12}$$

Using (10) and (13), eq. (A.12) yields an implicit functional relationship between λ and k . Differentiating (A.12) and evaluating at the steady state we get

$$\left[\frac{d\lambda}{dk} \right]_{k=0} = 0. \quad (\text{A.13})$$

From (6) and (1.) we get at the steady state

$$\frac{\partial \lambda}{\partial k} = -\lambda f_{kk} > 0,$$

$$\frac{\partial k}{\partial \lambda} = -\frac{f_h^2}{u_{hh} + \lambda f_{hh}} \frac{R_{bb}}{R_{bb} - f_{kk}} > 0. \quad (\text{A.14})$$

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